

Two-sided moment estimates for a class of nonnegative chaoses

Rafał Meller

Abstract

We derive two-sided bounds for moments of random multilinear forms (random chaoses) with nonnegative coefficients generated by independent nonnegative random variables X_i which satisfy the following condition on the growth of moments: $\|X_i\|_{2p} \leq A\|X_i\|_p$ for any i and $p \geq 1$. Estimates are deterministic and exact up to multiplicative constants which depend only on the order of chaos and the constant A in the moment assumption.

Keywords: Polynomial chaoses; Tail and moment estimates; Logarithmically concave tails.

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1 Introduction

In this paper we study homogeneous tetrahedral chaoses of order d , i.e. random variables of the form

$$S = \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} X_{i_1} \cdot \dots \cdot X_{i_d},$$

where X_1, \dots, X_n are independent random variables and (a_{i_1, \dots, i_d}) is a multiindexed symmetric array of real numbers such that $a_{i_1, \dots, i_d} = 0$ if $i_l = i_m$ for some $m \neq l$, $m, l \leq d$.

Chaoses of order $d = 1$ are just sums of independent random variables, object quite well understood. R. Łatała [5] derived two-sided bounds for $\|\sum a_i X_i\|_p$ under general assumptions that either a_i, X_i are nonnegative or X_i are symmetric. The case $d \geq 2$ is much less understood. There are papers presenting two-sided bounds for moments of S in special cases when (X_i) have normal distribution [6], have logarithmically concave tails [1] or logarithmically convex tails [3].

The purpose of this note is to derive two-sided bounds for $\|S\|_p$ if coefficients (a_{i_1, \dots, i_d}) are nonnegative and (X_i) are independent, nonnegative and satisfy the following moment condition for some $k \in \mathbb{N}$,

$$\|X_i\|_{2p} \leq 2^k \|X_i\|_p \quad \text{for every } p \geq 1. \quad (1)$$

The main idea is that if a r.v. X_i satisfy (1) then it is comparable with a product of k i.i.d. variables with logarithmically concave tails. This way the problem reduces to the result of Łatała and Łochowski [7] which gives two-sided bounds for moments of nonnegative chaoses generated by r.v.'s with logarithmically concave tails.

2 Notation and main results

We set $\|Y\|_p = (\mathbb{E}|Y|^p)^{1/p}$ for a real r.v. Y and $p \geq 1$, $\log(x) = \log_2(x)$ and \ln stands for the natural logarithm. By C, t_0 (sometimes $C(k, d), t_0(k, d)$) we denote constants, that

may depend on k, d and may vary from line to line. We write $A \sim_{k,d} B$ if $A \cdot C(k, d) \geq B$ and $B \cdot C(k, d) \geq A$.

Let $\{X_i^{(1)}\}, \dots, \{X_i^{(d)}\}$ be independent r.v's. We set

$$N_i^{(r)}(t) = -\ln \mathbb{P}(X_i^{(r)} \geq t).$$

We say that $X_i^{(r)}$ has logarithmically concave tails if the function $N_i^{(r)}$ is convex. We put

$$B_p^{(r)} = \left\{ v \in \mathbb{R}^n \mid \sum_{i=1}^n N_i^{(r)}(v_i) \leq p \right\}$$

and

$$\|(a_{i_1, \dots, i_d})\|_p = \sup \left\{ \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d (1 + v_{i_r}^{(r)}) \mid (v_i^{(r)}) \in B_p^{(r)} \right\}.$$

We will show the following

Theorem 2.1. *Let $(X_i^{(r)})_{r \leq d, i \leq n}$ be independent non-negative random variables satisfying (1) and $\mathbb{E}X_i^{(r)} = 1$. Then for any nonnegative coefficients $(a_{i_1, \dots, i_d})_{i_1, \dots, i_d \leq n}$ we have*

$$\frac{1}{C(k, d)} \|(a_{i_1, \dots, i_d})\|_p \leq \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} X_{i_1}^{(1)} \cdot \dots \cdot X_{i_d}^{(d)} \right\|_p \leq C(k, d) \|(a_{i_1, \dots, i_d})\|_p.$$

Theorem 2.1 in the same way as in the proof of Theorem 2.2 in [7] yields the following two-sided bounds for tails of random chaoses.

Theorem 2.2. *Under the assumptions of Theorem 2.1 there exist constants $0 < c(k, d), C(k, d) < \infty$ depending only on d and k such that for any $t \geq 0$ we have*

$$\mathbb{P} \left(\sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} X_{i_1}^{(1)} \cdot \dots \cdot X_{i_d}^{(d)} \geq C(k, d) \|(a_{i_1, \dots, i_d})\|_p \right) \leq e^{-t}$$

and

$$\mathbb{P} \left(\sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} X_{i_1}^{(1)} \cdot \dots \cdot X_{i_d}^{(d)} \geq c(k, d) \|(a_{i_1, \dots, i_d})\|_p \right) \geq \min(c(k, d), e^{-t}).$$

Now we are ready to present two-sided bounds for undecoupled chaoses. We define in this case $N_i(t) = -\ln \mathbb{P}(X_i \geq t)$,

$$B_p = \left\{ v \in \mathbb{R}^n \mid \sum_{i=1}^n N_i(v_i) \leq p \right\}$$

and

$$\|(a_{i_1, \dots, i_d})\|_p = \sup \left\{ \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d (1 + v_{i_r}^{(r)}) \mid (v_i^{(r)}) \in B_p \right\}.$$

Theorem 2.3. *Let $(X_i)_{i \leq n}$ be nonnegative independent r.v's satisfying (1) and $\mathbb{E}X_i = 1$. Then for any symmetric array of nonnegative coefficients $(a_{i_1, \dots, i_d})_{i_1, \dots, i_d \leq n}$ such that*

$$a_{i_1, \dots, i_d} = 0 \text{ if } i_l = i_m \text{ for some } m \neq l, \ m, l \leq d \quad (2)$$

we have

$$\frac{1}{C(k, d)} \|(a_{i_1, \dots, i_d})\|_p \leq \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} X_{i_1} \cdot \dots \cdot X_{i_d} \right\|_p \leq C(k, d) \|(a_{i_1, \dots, i_d})\|_p.$$

Moreover,

$$\begin{aligned} \mathbb{P} \left(\sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} X_{i_1} \cdot \dots \cdot X_{i_d} \geq C(k, d) \|(a_{i_1, \dots, i_d})\|_p \right) &\leq e^{-t}, \\ \mathbb{P} \left(\sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} X_{i_1} \cdot \dots \cdot X_{i_d} \geq c(k, d) \|(a_{i_1, \dots, i_d})\|_p \right) &\geq \min(c(k, d), e^{-t}). \end{aligned}$$

Proof. Let $S' = \sum a_{i_1, \dots, i_d} X_{i_1}^{(1)} \cdot \dots \cdot X_{i_d}^{(d)}$ be the decoupled version of $S = \sum a_{i_1, \dots, i_d} X_{i_1} \cdot \dots \cdot X_{i_d}$. By the results of de la Peña and Montgomery-Smith [2] (one may use also the result of Kwapien [4]) moments and tails of S and S' are comparable up to constants which depend only on d . Hence Theorem 2.3 follows by Theorems 2.1 and 2.2. \square

3 Preliminaries

In this section we study properties of nonnegative r.v's satisfying condition (1). We will assume normalization $\mathbb{E}X = 1$ and define $N(t) = -\ln \mathbb{P}(X \geq t)$.

Lemma 3.1. *There exists a constant $C = C(k)$ such that for any $x \geq 1$, and $t \geq 1$ we have $N(Ctx) \geq t^{\frac{1}{k}} N(x)$. One may take $C = 8^{k+1}$.*

Proof. We need to show that

$$\mathbb{P}(X \geq Ctx) \leq \mathbb{P}(X \geq x)^{t^{\frac{1}{k}}} \text{ for } t \geq 1, x \geq 1. \quad (3)$$

It is enough to prove the assertion for $x < \frac{\|X\|_\infty}{2}$ because for $x \geq \frac{\|X\|_\infty}{2}$, (3) is obvious for $C > 2$. In that case $x = \frac{1}{2} \|X\|_q$ for some $q \geq 1$ (since $\|X\|_1 = 1$). From the Paley-Zygmunt inequality and (1)

$$\begin{aligned} \mathbb{P}(X \geq x) &= \mathbb{P} \left(X^q \geq \frac{1}{2^q} \mathbb{E}X^q \right) \geq \left(1 - \frac{1}{2^q} \right)^2 \left(\frac{\|X\|_q}{\|X\|_{2q}} \right)^{2q} \\ &\geq \left(1 - \frac{1}{2^q} \right)^2 \frac{1}{2^{2kq}} \geq \left(\frac{1}{2^{k+1}} \right)^{2q}. \end{aligned} \quad (4)$$

Let $A = \lceil \frac{1}{k} \log(t) \rceil$. By (1) we get $\|X\|_{q^{2^A}} \leq 2^{kA} \|X\|_q$ hence Chebyshev's inequality yields

$$\mathbb{P}(X \geq Ctx) = \mathbb{P} \left(X \geq \frac{Ct}{2} \|X\|_q \right) \leq \mathbb{P} \left(X \geq \frac{Ct}{2^{kA+1}} \|X\|_{q^{2^A}} \right) \leq \left(\frac{2^{kA+1}}{Ct} \right)^{q^{2^A}}.$$

We have $2^A \geq t^{\frac{1}{k}}$ and $2^{kA} \leq 2^{k(\frac{1}{k} \log(t)+1)} = t2^k$ so if $C = 8^{k+1}$ then

$$\mathbb{P}(X \geq Ctx) \leq \left(\frac{1}{4^{k+1}} \right)^{qt^{\frac{1}{k}}} = \left(\frac{1}{2^{k+1}} \right)^{2qt^{\frac{1}{k}}}. \quad (5)$$

The assertion follows by (4) and (5) \square

In fact one may reverse the statement of Lemma 3.1.

Remark 3.2. Let X be a nonnegative r.v., $\mathbb{E}X = 1$ and there exist constants $C, \beta > 0$ such that $N(Ctx) \geq t^\beta N(x)$ for $t, x \geq 1$. Then there exists $\bar{K} = \bar{K}(C, \beta)$ such that

$$\|X\|_{2p} \leq \bar{K} \|X\|_p \quad \text{for } p \geq 1.$$

Proof. In this proof K means constant which may depend on C and β and vary from line to line. Integration by parts yields

$$\begin{aligned} \mathbb{E} \left| \frac{X}{2C} \right|^{2p} &= \int_0^\infty 2pt^{2p-1} e^{-N(2Ct)} dt \leq \|X\|_p^{2p} + \int_{\|X\|_p}^\infty 2pt^{2p-1} e^{-N(2Ct)} dt \\ &\leq \|X\|_p^{2p} + \int_{\|X\|_p}^\infty 2pt^{2p-1} e^{-N(2\|X\|_p)(\frac{t}{\|X\|_p})^\beta} dt. \end{aligned} \quad (6)$$

Let $\alpha = N(2\|X\|_p)^{\frac{1}{\beta}} / \|X\|_p$, substituting $y = \alpha t$ into (6) we get

$$\mathbb{E} \left| \frac{X}{2C} \right|^{2p} \leq \|X\|_p^{2p} + \frac{1}{\alpha^{2p}} \int_0^\infty 2py^{2p-1} e^{-y^\beta} dy.$$

Thus

$$\begin{aligned} \|X\|_{2p} &\leq 2C \|X\|_p + \frac{2C}{\alpha} \left(\int_0^\infty 2py^{2p-1} e^{-y^\beta} dy \right)^{1/2p} \\ &= 2C \|X\|_p + \frac{2C}{\alpha} \left(\frac{2p}{\beta} \Gamma\left(\frac{2p}{\beta}\right) \right)^{1/2p} \leq 2C \|X\|_p + K \frac{p^{\frac{1}{\beta}}}{\alpha}. \end{aligned} \quad (7)$$

By Chebyshev's inequality $N(2\|X\|_p) \geq p \ln 2$ and the assertion follows by (7). \square

Now we state the crucial technical lemma.

Lemma 3.3. There exists C, c, t_0 which depend on k , a probability space with a version of X and nonnegative i.i.d. r.v's Y_1, \dots, Y_k with the following properties

- (i) $C(X + t_0) \geq Y_1 \cdot \dots \cdot Y_k$,
- (ii) $C(Y_1 \cdot \dots \cdot Y_k + t_0) \geq X$,
- (iii) Y_1, \dots, Y_k have log-concave tails,
- (iv) $H(t) \leq N(t^k) \leq H(Ct)$ for $t \geq t_0$, where $H(t) = -\ln \mathbb{P}(Y_l \geq t)$,
- (v) $\frac{1}{C} \leq \mathbb{E}Y_l \leq C$.

Proof. Let $M(t) = N(t^k)$. By Lemma 3.1 there exists C (depending on k) such that $M(C\lambda t) \geq \lambda M(t)$ for all $\lambda \geq 1, t \geq 1$. By [8, Lemma 3.5] (applied with $t_0 = 1$) there exists convex nondecreasing function H , constants $C = C(k), t_0 = t_0(k) > 0$ such that

$$\begin{aligned} H(t) &\leq M(t) \leq H(Ct) \quad \text{for } t \geq t_0 \\ H(t) &= 0 \quad \text{for } t \leq t_0 \end{aligned} \quad (8)$$

Let Y_i be nonnegative i.i.d. r.v's such that $\mathbb{P}(Y_l \geq t) = e^{-H(t)}$, then (iii) and (iv) hold.

Now we verify (i) and (ii). For $t \geq \max\{1, t_0\}$ we have

$$\begin{aligned} \mathbb{P} \left(\prod_{l=1}^k Y_l \geq t \right) &\geq \prod_{l=1}^k \mathbb{P} \left(Y_l \geq t^{\frac{1}{k}} \right) = e^{-kH(t^{\frac{1}{k}})} \geq e^{-kM(t^{\frac{1}{k}})} = e^{-kN(t)} \\ &\geq e^{-N(Ck^k t)} = \mathbb{P}(X \geq Ck^k t), \end{aligned} \quad (9)$$

where the last inequality comes from Lemma 3.1. Furthermore,

$$\mathbb{P}\left(\prod_{l=1}^k Y_l \geq C^k t\right) \leq \sum_{l=1}^k \mathbb{P}(Y_l \geq C t^{\frac{1}{k}}) = k e^{-H(C t^{\frac{1}{k}})} \leq k e^{-M(t^{\frac{1}{k}})} = k e^{-N(t)}. \quad (10)$$

By Chebyshev's inequality $1 = \mathbb{E}X \geq e \mathbb{P}(X \geq e) = e^{1-N(e)}$, so $N(e) \geq 1$ and by Lemma 3.1 we get for $t \geq 1$, $N(Cte) \geq t^{\frac{1}{k}} N(e) \geq t^{\frac{1}{k}}$. Thus

$$\ln k - N(t) \leq -\frac{1}{2}N(t) \quad \text{for } t \geq eC \max(1, 2 \ln k)^k. \quad (11)$$

Lemma 3.1 also gives $\frac{N(t)}{2} \geq N(\frac{t}{2^k C})$ for $t > 2^k C$, so from (11) and (10)

$$\mathbb{P}\left(\prod_{l=1}^k Y_l \geq C^k t\right) \leq e^{-N(\frac{t}{2^k C})} = \mathbb{P}\left(X \geq \frac{t}{2^k C}\right). \quad (12)$$

Inequalities (9) and (12) implies (i) and (ii). To show (v) observe that

$$(\mathbb{E}Y_l)^k = \mathbb{E}Y_1 \cdot \dots \cdot Y_k \leq C(\mathbb{E}X + t_0) = C(1 + t_0)$$

an by (8)

$$\mathbb{E}Y_l \geq t_0 > 0$$

□

4 Proof of Theorem

Let $X_i^{(r)}$, $r \leq d, i \leq n$ satisfy the assumptions of Theorem 2.1. By Lemma 3.3 we may assume (enlarging if necessary the probability space) that there exist independent r.v's $Y_{i,l}^{(r)}$, $l \leq k, r \leq d, i \leq n$ such that conditions (i)-(v) of Lemma 3.3 hold (for $X_i^{(r)}$ and $Y_{i,l}^{(r)}$ instead of X and Y_l). Let $H_i^{(r)}(x) := -\ln \mathbb{P}(Y_{i,l}^{(r)} \geq x)$ (observe that this function does not depend on l).

Let us start with the following Proposition.

Fact 4.1. *For any $p \geq 1$,*

$$\begin{aligned} \frac{1}{C(k, d)} \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d X_{i_r}^{(r)} \right\|_p &\leq \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k Y_{i_r, l}^{(r)} \right\|_p \\ &\leq C(k, d) \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d X_{i_r}^{(r)} \right\|_p. \end{aligned}$$

Proof. Lemma 3.3 (ii) yields

$$\begin{aligned} &\left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d X_{i_r}^{(r)} \right\|_p \\ &\leq C^d \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k (Y_{i_r, l}^{(r)} + t_0) \right\|_p \\ &\leq C^d \sum_{\substack{\varepsilon_r \in \{0, 1\} \\ r=1, \dots, d}} \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k \left((Y_{i_r, l}^{(r)})^{\varepsilon_r} t_0^{1-\varepsilon_r} \right) \right\|_p. \end{aligned} \quad (13)$$

We have $\mathbb{E}Y_{i,l}^{(r)} \geq \frac{1}{C}$, so by Jensen's inequality we get for any $\varepsilon \in \{0, 1\}^d$,

$$\begin{aligned} \left\| \sum a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k Y_{i_r, l}^{(r)} \right\|_p &\geq \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k (Y_{i_r, l}^{(r)})^{\varepsilon_r} (\mathbb{E}Y_{i_r, l}^{(r)})^{1-\varepsilon_r} \right\|_p \\ &\geq \frac{1}{C^{kd}} \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k \left((Y_{i_r, l}^{(r)})^{\varepsilon_r} (t_0)^{1-\varepsilon_r} \right) \right\|_p. \end{aligned} \quad (14)$$

The lower estimate in Proposition 4.1 follows by (13) and (14). The proof of the upper bound is analogous. \square

So to prove Theorem 2.1 we need to estimate $\left\| \sum a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k Y_{i_r, l}^{(r)} \right\|_p$. To this end we will apply the following result of Latała and Łochowski.

Theorem 1 ([7, Theorem 2.1]). *Let $\{Z_i^{(1)}\}, \dots, \{Z_i^{(d)}\}$ be independent nonnegative r.v.'s with logarithmically concave tails and $M_i^{(r)}(t) = -\ln \left(\mathbb{P} \left(Z_i^{(r)} \geq t \right) \right)$. Assume that $1 = \inf\{t > 0 : M_i^{(r)}(t) \geq 1\}$. Then*

$$\begin{aligned} &\frac{1}{C} \sup \left\{ \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d (1 + b_{i_r}^{(r)}) \mid (b_i^{(r)}) \in T_p^{(r)} \right\} \\ &\leq \left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} Z_{i_1}^{(1)} \dots Z_{i_d}^{(d)} \right\|_p \\ &\leq C \sup \left\{ \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d (1 + b_{i_r}^{(r)}) \mid (b_i^{(r)}) \in T_p^{(r)} \right\}, \end{aligned}$$

where $T_p^{(r)} = \left\{ b \in \mathbb{R}_+^n : \sum_{i=1}^n M_i^{(r)}(b_i) \leq p \right\}$.

To use the above result we need to normalize variables $Y_{i,l}^{(r)}$. Let

$$t_i^{(r)} = \inf\{t > 0 : H_i^{(r)}(t) \geq 1\}, \quad r \leq d, \quad i \leq k.$$

Lemma 3.3 (v) gives $t_i^{(r)} \leq e\mathbb{E}Y_{i,l}^{(r)} \leq eC$ and by (8) $t_i^{(r)} \geq t_0 > 0$, thus

$$\frac{1}{C(k, d)} \leq t_i^{(r)} \leq C(k, d). \quad (15)$$

Theorem applied to variables $Y_{i,l}^{(r)} = Y_{i,l}^{(r)} / t_i^{(r)}$ together with (15) gives

$$\begin{aligned} &\left\| \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k Y_{i_r, l}^{(r)} \right\|_p \\ &\sim_{k,d} \sup \left\{ \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k \left(1 + v_{i_r, l}^{(r)} \right) \mid (v_{i,l}^{(r)}) \in D_{k,p}^{(r)}, \quad r = 1, \dots, d \right\} \end{aligned}$$

where

$$D_{k,p}^{(r)} = \left\{ (v_{i,l})_{i \leq n; l \leq k} \in \mathbb{R}^{nk} : \sum_{i=1}^n H_i^{(r)}(v_{i,l}) \leq p \text{ for all } l \leq k \right\}.$$

Lemma 3.3 (iv) yields

$$\begin{aligned} & \sup \left\{ \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k \left(1 + v_{i_r, l}^{(r)} \right) \mid \left(v_{i, l}^{(r)} \right) \in D_{k, p}^{(r)}, r = 1, \dots, d \right\} \\ & \sim_{k, d} \sup \left\{ \sum_{1 \leq i_1, \dots, i_d \leq n} a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k \left(1 + v_{i_r, l}^{(r)} \right) \mid \left(v_{i, l}^{(r)} \right) \in B_{k, p}^{(r)}, r = 1, \dots, d \right\} \\ & =: \|(a_{i_1, \dots, i_d})\|'_{k, p}, \end{aligned}$$

where

$$B_{k, p}^{(r)} = \left\{ (v_{i, l})_{i \leq n, l \leq k} \in \mathbb{R}^{nk} : \sum_{i=1}^n N_i^{(r)}(v_{i, l}^k) \leq p \text{ for all } l \leq k \right\}.$$

To finish the proof of Theorem 2.1 we need to show that

$$\|(a_{i_1, \dots, i_d})\|'_{k, p} \sim \|(a_{i_1, \dots, i_d})\|_p. \quad (16)$$

First we will show this holds for $d = 1$, that is

$$\begin{aligned} & \sup \left\{ \sum b_i \prod_{l=1}^k (1 + a_{i, l}) \mid \sum N_i(a_{i, l}^k) \leq p \text{ for all } l \leq k \right\} \\ & \sim_k \sup \left\{ \sum b_i (1 + w_i) \mid \sum N_i(w_i) \leq p \right\}. \end{aligned} \quad (17)$$

We have

$$\begin{aligned} & \sup \left\{ \sum b_i \prod_{l=1}^k (1 + a_{i, l}) \mid \sum N_i(a_{i, l}^k) \leq p \text{ for all } l \leq k \right\} \\ & \leq \sum_{\substack{\varepsilon_l \in \{0, 1\} \\ l=1 \dots k}} \sup \left\{ \sum b_i \prod_{l=1}^k a_{i, l}^{\varepsilon_l} \mid \sum N_i(a_{i, l}^k) \leq p \text{ for all } l \leq k \right\}. \end{aligned}$$

So to establish the upper bound in (17) it is enough to prove

$$\begin{aligned} & \sup \left\{ \sum b_i \prod_{l=1}^k a_{i, l}^{\varepsilon_l} \mid \sum N_i(a_{i, l}^k) \leq p \text{ for all } l \leq k \right\} \\ & \leq C(k) \sup \left\{ \sum b_i (1 + w_i) \mid \sum N_i(w_i) \leq p \right\} \end{aligned}$$

or equivalently (after permuting indexes) that for any $0 \leq k_0 \leq k$,

$$\begin{aligned} & \sup \left\{ \sum b_i \prod_{l=1}^{k_0} a_{i, l} \mid \sum N_i(a_{i, l}^k) \leq p \text{ for all } l \leq k_0 \right\} \\ & \leq C(k) \sup \left\{ \sum b_i (1 + w_i) \mid \sum N_i(w_i) \leq p \right\}. \end{aligned} \quad (18)$$

Let us fix sequences $(a_{i, l})$ such that $\sum N_i(a_{i, l}^k) \leq p$ for all $l \leq k_0$. Let C be a constant from Lemma 3.1, define

$$w_i = \begin{cases} \frac{\prod_{l=1}^{k_0} a_{i, l}}{Ck^k} & \text{if } \prod_{l=1}^{k_0} a_{i, l} > 2Ck^k \\ 0 & \text{otherwise} \end{cases}$$

For such w_i we have $\prod_{l=1}^{k_0} a_{i,l} \leq 2Ck^k(1+w_i)$, so to establish (18) it is enough to check that $\sum N_i(w_i) \leq p$. Lemma 3.1 however yields

$$\begin{aligned} \sum_i N_i(w_i) &\leq \frac{1}{k} \sum_i N_i(Ck^k \cdot w_i) \leq \frac{1}{k} \sum_{i: w_i \neq 0} N_i(\max\{a_{i,1}, \dots, a_{i,k_0}\}^{k_0}) \\ &\leq \frac{1}{k} \sum_i N_i(\max\{a_{i,1}, \dots, a_{i,k_0}\}^k) \leq \frac{1}{k} \sum_i \sum_{l=1}^{k_0} N_i(a_{i,l}^k) \leq p. \end{aligned}$$

where the third inequality comes from the observation that $w_i \neq 0$ implies $\max\{a_{i,1}, \dots, a_{i,k_0}\} \geq 1$.

To show the lower bound in (17) we fix $w_i \in B_p$, choose $a_{i,1} = a_{i,2} = \dots = a_{i,k} = w_i^{\frac{1}{k}}$ and observe that

$$\sum b_i \prod_{l=1}^k (1 + a_{i,l}) = \sum b_i \left(1 + w_i^{\frac{1}{k}}\right)^k \geq \sum b_i (1 + w_i).$$

We showed that (17) holds. Now we prove (16) for any d . We have

$$\begin{aligned} \|(a_{i_1, \dots, i_d})\|'_{k,p} &= \sup \left\{ \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k \left(1 + v_{i_r, l}^{(r)}\right) \mid \left(v_{i_r, l}^{(r)}\right) \in B_{k,p}^{(r)}, r \leq d \right\} \\ &= \sup \left\{ \sup \left\{ \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} \prod_{r=1}^d \prod_{l=1}^k \left(1 + v_{i_r, l}^{(r)}\right) \mid (v_{i_r, l}^{(d)}) \in B_{k,p}^{(d)} \right\} \mid \forall_{r \leq d-1} (v_{i_r, j}^{(r)}) \in B_{k,p}^{(r)} \right\} \\ &\sim_k \sup \left\{ \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} \prod_{r=1}^{d-1} \prod_{l=1}^k \left(1 + v_{i_r, l}^{(r)}\right) \left(1 + w_{i_d}^{(d)}\right) \mid (w^{(d)}) \in B_p^{(d)}, \forall_{r \leq d-1} v_{i_r, j}^{(r)} \in B_{k,p}^{(r)} \right\}, \end{aligned} \tag{19}$$

where the last equivalence follows by (17). Iterating the above procedure d times we obtain (16).

Remark 4.2. *Deeper analysis of the proof shows that constant C from Theorem 2.1 is less than $(C')^{k^3 d} k^{6k^2 d} d^{kd}$ for C' a universal constant.*

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Rafał Meller
 Institute of Mathematics
 University of Warsaw
 Banacha 2, 02-097 Warszawa, Poland
r.meller@mimuw.edu.pl